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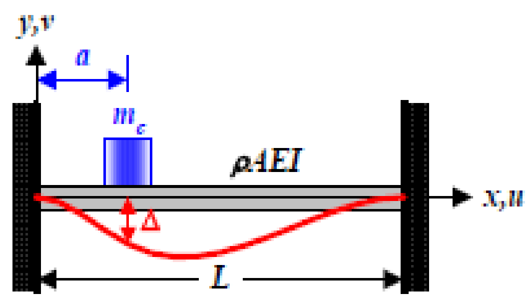
CITY UNIVERSITY OF NEW YORK

HOMEWORK #2



EQUIVALENT SPRING AND MASS (FIXED-FIXED BEAM)

ME 411: System Modeling Analysis and Control

Fall 2010

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September 29, 2010

Abstract

In mechanical system design, mathematical modeling of dynamic systems is an essential tool for the study of the physics behind the study of the problem. In this problem, a concentrated mass is mounted on a fixed-fixed beam at a distance (a) from the left fixed end. The beam has a density (ρ), a Young's modulus (E), a uniform cross-sectional area (A), a moment of inertia (I) and a length (L). The displacements, u and v, along the Cartesian coordinates are represented in, x and y, respectively. However, that the boundary conditions at a fixed (or clamped or built-in) end are prescribed deflection and slope. Analytical method was used for solving ODEs developed for the system of the statically indeterminate system of beam. At the end, the reactions and moment at the ends, the shear moment diagram of the system and the equivalent spring mass approach was used to solve for the system of linear algebraic equations with numerical software MATLAB.

1.0 Nomenclature

$\boldsymbol{a = distance\ of\ loading\ from\ the\ origin\ (x = 0)}$

$\boldsymbol{L = total\ length\ of\ the\ beam}$

$\boldsymbol{\rho = density}$

$\boldsymbol{A = cross\ sectional\ area}$

$\boldsymbol{I = moment\ of\ inertia}$

$\boldsymbol{E = young\ modulus}$

$\boldsymbol{m_c = mass\ of\ the\ load}$

$\boldsymbol{m_e = mass\ of\ spring}$

$\boldsymbol{K_e = spring\ constant}$

$\boldsymbol{\theta(x) = slope\ of\ the\ deflected\ beamas\ the\ fucntion\ of\ length}$

$\boldsymbol{v(x) = deflectionas\ the\ fucntion\ of\ length}$

$\boldsymbol{M(x) = bending\ moment}$

$\boldsymbol{V(x) = shearforce}$

1. Background

A mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms. In very general sense, it can be represented as a functional relationship of the form

$$\boldsymbol{Dependent\ Variable = f\begin{pmatrix} independent\ variables, \\ parameters, \quad forcing\ fucntions \end{pmatrix}}$$

where, the dependent variable is a characteristic that usually reflects the behavior or state of the system. Independent variables are usually dimensions, such as time and space, the parameters are reflective of the systems properties or composition and the forcing functions are external influences acting upon the system.

2. Theory

The deflection of the beam, in particular the maximum deflection of the beam under given loading is required to be analyze, in particular the knowledge is deflection of the beam is required to analyze the statically indeterminate beam as the number of reactions at the supports exceeds the number of equilibrium equations available to determine these unknowns.

The deformation of the member caused by the bending moment M is measured by the curvature of the neutral surface. The curvature (c) is defined as the reciprocal of the radius of the curvature (φ) and can be written as

$$\frac{1}{\varphi} = \frac{\epsilon_{max}}{c}$$

But in elastic range, we have $\epsilon_{max} = \frac{\sigma_{max}}{E}$

$$\frac{1}{\varphi} = \frac{1}{c}\frac{\sigma_{max}}{E}$$

In case of pure bending, the neutral axis passes through the centroid of the cross section,

$$\sigma_{max} = \frac{Mc}{I}$$

Therefore,

$$\frac{1}{\varphi} = \frac{1}{c}\frac{1}{E}\frac{Mc}{I} = \frac{M}{IE} - - - - - - - - -(a)$$

We have, the curvature of the a plane at a given point p(x, y) of a curve can be expressed as

$$\frac{1}{\varphi} = \frac{\left(\frac{d^2y}{dx^2}\right)}{\left[1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

Where, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ are the first and second derivatives of the function y(x) represented by the curve. But, in the case of the elastic curve of abeam, the slope $\frac{dy}{dx}$ is very small and its square is negligible compared to unity, we write, therefore,

$$\frac{1}{\varphi} = \frac{d^2y}{dx^2}$$

Therefore, in the equations--------- (a)

$$\frac{d^2y}{dx^2} = \frac{M(x)}{IE} - - - - - - - - - - - - (b)$$

In mathematical expression, the equation obtain is a second order linear differential equations, governing the equation for the case if the elastic curve.

3. Statically determinate vs. statically indeterminate

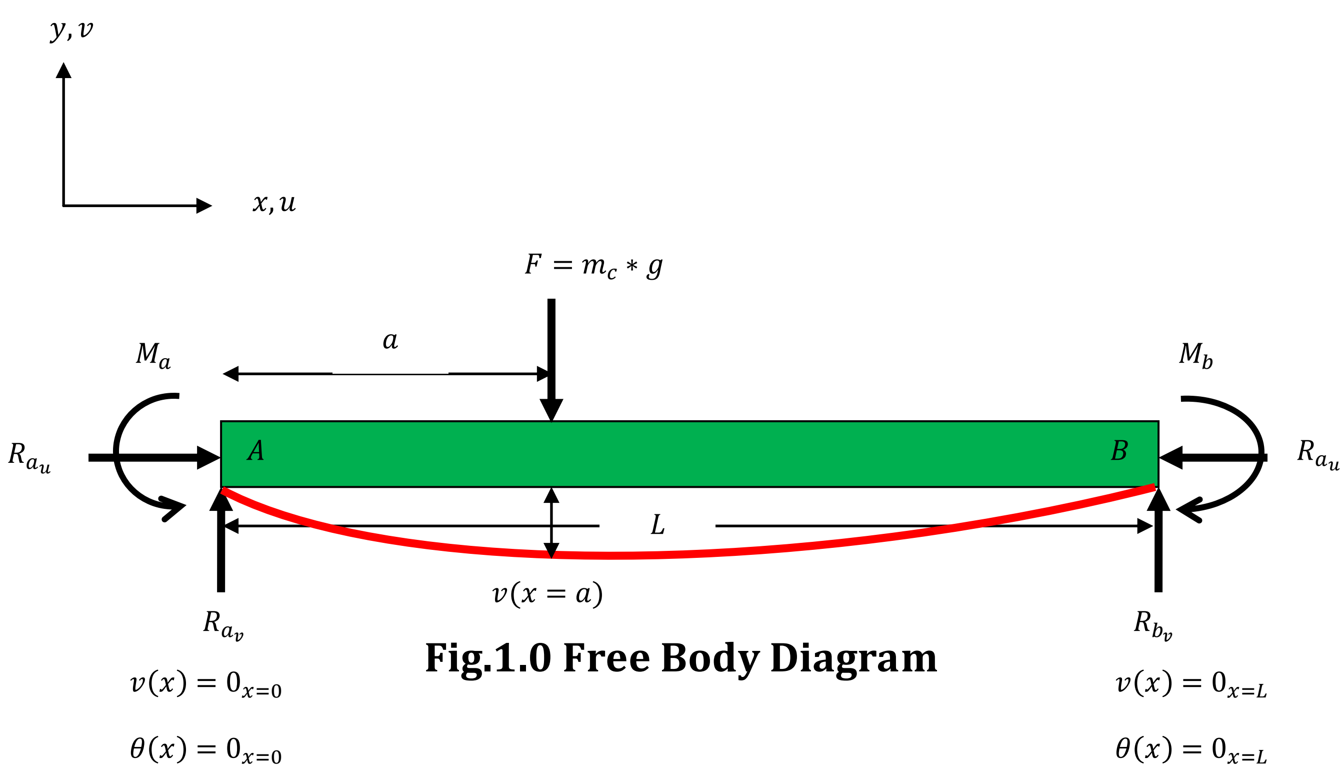


Fig.1.0 Free Body Diagram

There are six unknowns, $R_{a_v}, R_{a_u}, R_{b_u}, R_{b_v}, M_a, M_b$, while only three equilibrium equations are available.

From free body diagram above, writing the equations of equilibrium,

$$\sum F_v = 0, \sum F_u = 0 \; and \; \sum M = 0$$

Let us write the equations,

$$\uparrow + \sum F_v = 0 \rightarrow R_{a_v} + R_{b_v} = F$$

$$\rightarrow + \sum F_u = 0 \rightarrow R_{a_u} + R_{b_u} = 0 \rightarrow R_{a_u} = -R_{b_u}$$

$$anticlockwise(+) \sum M_A = 0 \rightarrow R_{b_v} = \frac{M_b - M_a + F * a}{L}$$

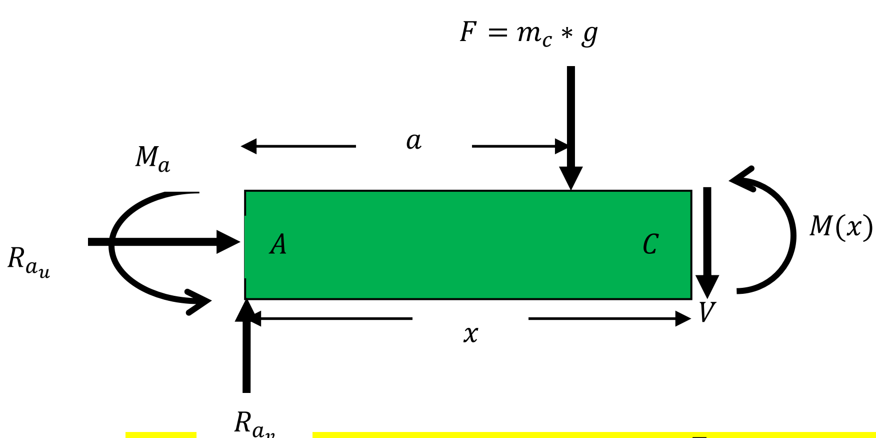
Therefore,

$$R_{a_v} = \frac{F * L - M_b + M_a - F * a}{L}$$

Since, from these equations we can't measure all the unknowns; hence we conclude the beam is statically indeterminate.

4. ODE method for calculate deflection, slope, bending moment, shear force

Taking small section of the beam on left side



$$\uparrow + \sum F_v = 0 \rightarrow V(x) = \frac{F * L - M_b + M_a - F * a}{L} - F < x - a >^0 - - - - (c)$$

$$anticlockwise(+) \sum M_c = 0 \rightarrow M(x) = \frac{F * L - M_b + M_a - F * a}{L} * x - F < x - a >^1 - M_a$$

And from the FBD: $R_{a_u} = R_{b_u} = 0$

$This\ is\ a\ step\ fucntion$: $< x-a >^0 = \begin{Bmatrix} 1 & x \geq a \\ 0 & x < a \end{Bmatrix}$ and $< x-a >^n = \begin{Bmatrix} (x-a)^n & x \geq a \\ 0 & x < a \end{Bmatrix}$

From the equation number (b)

$$\frac{d^2y}{dx^2} = \frac{M(x)}{IE}$$

Integrating the equation three times and adding constants,

$$EI\frac{d^2y}{dx^2} = \frac{F*L - M_b + M_a - F*a}{L}*x - F < x-a >^1 - M_a$$

$$EI\frac{dy}{dx} = \theta(x) = \frac{1}{2}\frac{F*L - M_b + M_a - F*a}{L}*x^2 - \frac{F}{2} < x-a >^2 - M_a*x + C_1$$

$$EI\,y = v(x) = \frac{1}{6}\frac{F*L - M_b + M_a - F*a}{L}*x^3 - \frac{1}{6}F < x-a >^3 - \frac{1}{2}M_a*x^2 + C_1x + C_2$$

Solving with first boundary condition and then from equation of equilibrium:

$v(x) = 0_{x=0}$

$\theta(x) = 0_{x=0}$

$C_1 = C_2 = 0$

And also at

$v(x) = 0_{x=L}$

$\theta(x) = 0_{x=L}$

From these, solving via Maple 13

$$solve\left(\left[\frac{(-M_b + M_a - F\cdot a + F\cdot L)}{2\cdot L}\cdot L^2 - \frac{F}{2}\cdot(L-a)^2 - M_a\cdot L, \frac{(-M_b + M_a - F\cdot a + F\cdot L)}{6\cdot L}\cdot L^3 - \frac{F}{6}\cdot(L-a)^3 - \frac{M_a}{2}\cdot L^2\right], \{M_a, M_b\}\right)$$

$$\left\{M_a = \frac{F\,a\,(L^2 - 2\,L\,a + a^2)}{L^2}, M_b = \frac{a^2\,F\,(L-a)}{L^2}\right\}$$

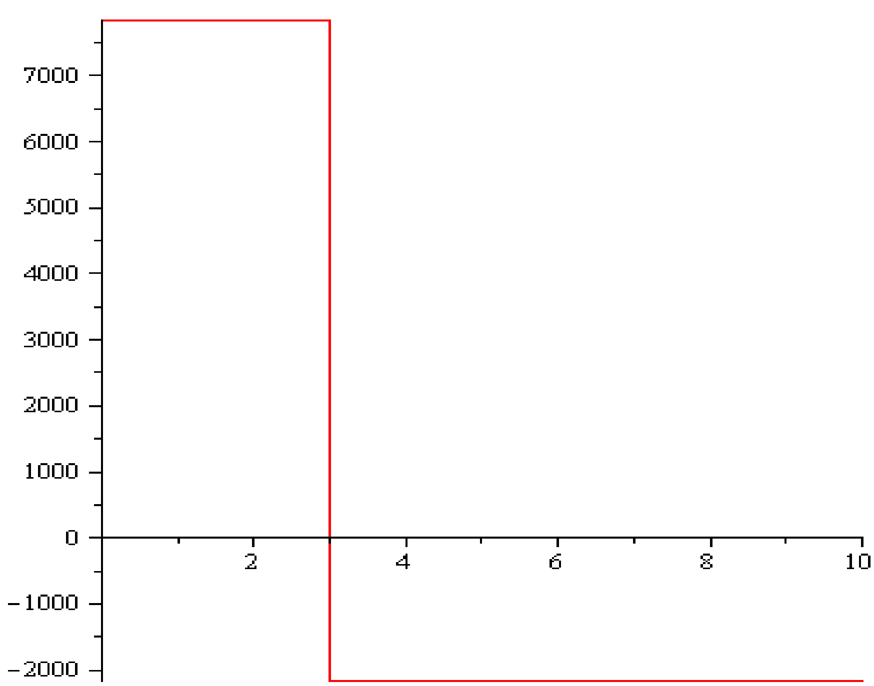
$$R_{a_v} = \frac{\frac{a^2\,F\,(L-a)}{L^2} - \frac{F\,a\,(L^2 - 2\,L\,a + a^2)}{L^2} + F\,a}{L} \qquad R_{a_v} = \frac{-\frac{a^2\,F\,(L-a)}{L^2} + \frac{F\,a\,(L^2 - 2\,L\,a + a^2)}{L^2} - F\,a + F\,L}{L}$$

$$0 \le x < a \left\{ \begin{array}{c} 1 \\ Shear\ force: V(x) = \frac{F*L - M_b + M_a - F*a}{L} \\ Bending\ Moment: \frac{d^2y}{dx^2} = M(x) = \frac{1}{EI}\left(\frac{F*L - M_b + M_a - F*a}{L} * x - M_a\right) \\ Slope: \frac{dy}{dx} = \theta(x) = \frac{1}{EI}\left(\frac{1}{2}\frac{F*L - M_b + M_a - F*a}{L} * x^2 - M_a * x\right) \\ Deflection:.. v(x) = \frac{1}{EI}\left(\frac{1}{6}\frac{F*L - M_b + M_a - F*a}{L} * x^3 - \frac{1}{2}M_a * x^2\right) \end{array} \right\}$$

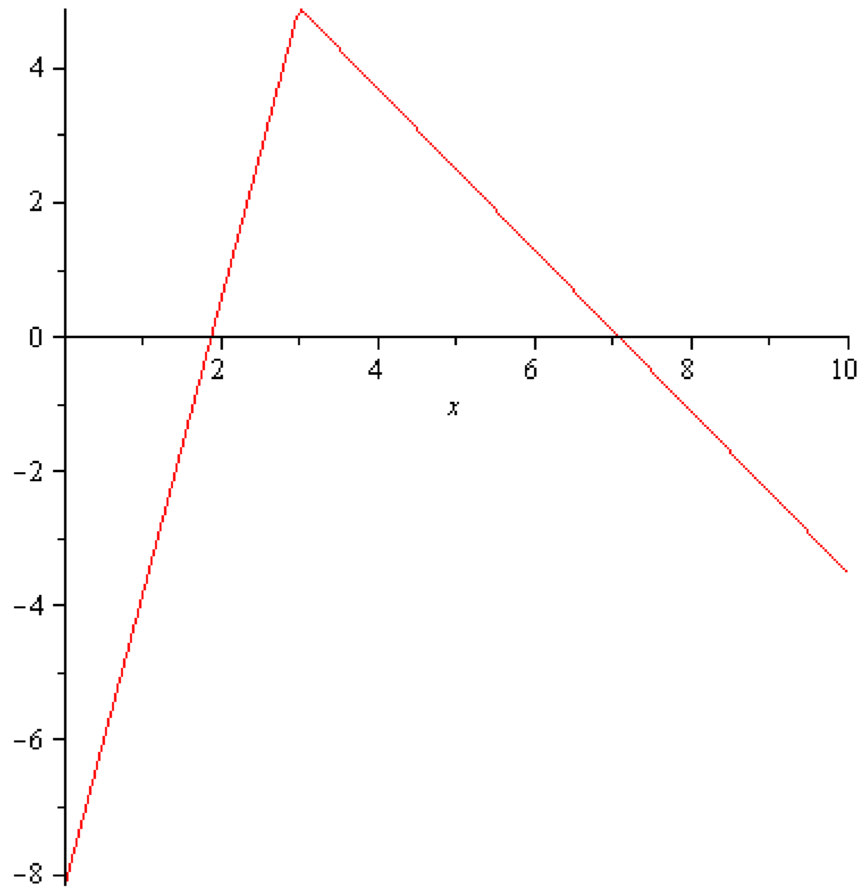
$$a \le x \le L \left\{ \begin{array}{c} 2 \\ Shear\ force: V(x) = \frac{F*L - M_b + M_a - F*a}{L} - F \\ Bending\ Moment: \frac{d^2y}{dx^2} = M(x) = \frac{1}{EI}(\frac{F*L - M_b + M_a - F*a}{L} * x - F(x-a) - M_a) \\ Slope: \frac{dy}{dx} = \theta(x) = \frac{1}{EI}\left(\frac{1}{2}\frac{F*L - M_b + M_a - F*a}{L} * x^2 - \frac{F}{2}(x-a)^2 - M_a * x\right) \\ Deflection:.. v(x) = \frac{1}{EI}\left(\frac{1}{6}\frac{F*L - M_b + M_a - F*a}{L} * x^3 - \frac{1}{6}F(x-a)^3 - \frac{1}{2}M_a * x^2\right) \end{array} \right\}$$

5. Shear moment diagram for $(\mathbf{0} \le \boldsymbol{X} \le \boldsymbol{L})$

$plot(shear(x), x = 0..L)$



plot(moment(x), x = 0..L)



6. **Deflection and slope at X=a.**

Deflection:

$$-\left(\frac{1}{3}\right)\frac{F*a^3(3*a^2*L-a^3+L^3-3*a*L^2)}{L^3*E*In}$$

Slope:

$$-\left(\frac{1}{2}\right)\frac{a^2*F*(5*a^2*L-2*a^3+L^3-4*a*L^2)}{L^3*E*In}$$

7. Equivalent spring mass and spring constant.

The dynamics of the beam mass system can be views as an equivalent mass- spring system:

$$m_t\ddot{\Delta}(\mathrm{t}) + \mathrm{K_e}\Delta(\mathrm{t}) = 0 \qquad \rightarrow m_t = \mathrm{m_c} + \mathrm{m_e}$$

Where,

$\Delta(\mathrm{t}) =$ time history of the deflectiona at the given locationwhere the mass is mounted

$\mathrm{m_e} =$ mass of th beam $= \rho\mathrm{AL}$

$\mathrm{K_e} =$ equivalent spring constant

$\mathrm{m_c} =$ concentric load

$\mathrm{m_c} =$ total load

(a) $\mathrm{K_e}@\, x = a$

$\Delta = -\mathrm{v}(0) = \mathrm{v}(\mathrm{x})|_{\mathrm{x=a}}$

$$\Delta = \frac{\left(-\frac{1}{6}\right) * \left(-\mathrm{a}^2 * \mathrm{F} * \frac{\mathrm{L-a}}{\mathrm{L}^2} + \mathrm{F} * \mathrm{a} * \frac{\mathrm{L}^2 - 2*\mathrm{L}*\mathrm{a} + \mathrm{a}^2}{\mathrm{L}^2} - \mathrm{F} * \mathrm{a} + \mathrm{F} * \mathrm{L}\right) * \frac{\mathrm{a}^3}{\mathrm{L}} + \left(\frac{1}{2}\right) * \mathrm{F} * \mathrm{a}^3 * \frac{\mathrm{L}^2 - 2*\mathrm{L}*\mathrm{a} + \mathrm{a}^2}{\mathrm{L}^2}}{\mathrm{E} * \mathrm{In}}$$

$$solve\left(\Delta = \frac{1}{E\,In}\left(-\frac{1}{6}\,\frac{1}{L}\left(\left(-\frac{a^2\,F\,(L-a)}{L^2} + \frac{F\,a\,(L^2 - 2\,L\,a + a^2)}{L^2} - F\,a + F\,L\right) a^3\right) + \frac{1}{2}\,\frac{F\,a^3\,(L^2 - 2\,L\,a + a^2)}{L^2}\right), F\right)$$

$$\mathrm{F} = \frac{3 * \Delta * \mathrm{L}^3 * \mathrm{E} * \mathrm{In}}{\mathrm{a}^3 * (3 * \mathrm{a}^2 * \mathrm{L} - \mathrm{a}^3 + \mathrm{L}^3 - 3 * \mathrm{a} * \mathrm{L}^2)}$$

Consider the clamped-clamped beam as a spring oscillating under the action of the concentrated mass. From Hooke's law: $\mathrm{F} = \mathrm{K_e}\Delta$, the equivalent spring constant should be:

$$\mathrm{K_e}\cancel{\Delta} = \frac{3 * \cancel{\Delta} * \mathrm{L}^3 * \mathrm{E} * \mathrm{In}}{\mathrm{a}^3 * (3 * \mathrm{a}^2 * \mathrm{L} - \mathrm{a}^3 + \mathrm{L}^3 - 3 * \mathrm{a} * \mathrm{L}^2)}$$

$$\mathrm{K_e} = \frac{3 * \mathrm{L}^3 * \mathrm{E} * \mathrm{In}}{\mathrm{a}^3 * (3 * \mathrm{a}^2 * \mathrm{L} - \mathrm{a}^3 + \mathrm{L}^3 - 3 * \mathrm{a} * \mathrm{L}^2)}$$

(a) $K_e @ x = a = \frac{L}{2}$

$$\Delta = -v(0) = v(x)|_{x=a=\frac{L}{2}}$$

$$\Delta = \frac{\left(\frac{1}{48} * \left(-a^2 * F * \frac{L-a}{L^2} + F * a * \frac{L^2-2*L*a+a^2}{L^2} - F * a + F * L\right)\right) * L^2 - \left(\frac{1}{6}\right) * F * \left(\left(\frac{1}{2}\right) * L - a\right)^3 - \left(\frac{1}{8}\right) * F * a * (L^2 - 2 * L * a + a^2)}{E * In}$$

$$solve\left(\Delta = \frac{1}{E\,In}\left(-\frac{1}{48}\left(-\frac{a^2 F (L-a)}{L^2} + \frac{F a (L^2 - 2 L a + a^2)}{L^2} - F a + F L\right) L^2 + \frac{1}{6} F \left(\frac{1}{2} L - a\right)^3 + \frac{1}{8} F a (L^2 - 2 L a + a^2)\right), F\right)$$

$$F = \frac{-192 * \Delta * E * In}{L^3}$$

Consider the clamped-clamped beam as a spring oscillating under the action of the concentrated mass. From Hooke's law: $F = K_e \Delta$, the equivalent spring constant should be:

$$K_e \cancel{\Delta} = \frac{-192 * \cancel{\Delta} * E * In}{L^3}$$

$$K_e = \frac{-192 * E * In}{L^3}$$

Following the justification of the elastic curve, the deflection is given by,

$$0 \le x < a\left\{Deflection:.\,v(x) = \frac{1}{EI}\left(\frac{1}{6}\frac{F * L - M_b + M_a - F * a}{L} * x^3 - \frac{1}{2} M_a * x^2\right)\right\}$$

$$a \le x \le L\left\{Deflection:.\,v(x) = \frac{1}{EI}\left(\frac{1}{6}\frac{F * L - M_b + M_a - F * a}{L} * x^3 - \frac{1}{6} F (x-a)^3 - \frac{1}{2} M_a * x^2\right)\right\}$$

Solving v(x) for a=L/2

$$\left(\frac{-\frac{16*\Delta*E*In*x^3}{L^3}+\frac{12*\Delta*E*In*x^2}{L^2}}{E*In}\right) 0 \le x < \frac{L}{2}$$

$$\left(\frac{-\frac{16*\Delta*E*In*x^3}{L^3}+\frac{32*\Delta*E*In*\left(x-\frac{L}{2}\right)^3}{L^3}+\frac{12*\Delta*E*In*x^2}{L^2}}{E*In}\right) \frac{L}{2} \le x \le L$$

Assume that during vibration the form of this beam deflection (i.e., the elastic curve) is preserved (Justification?), then:

$$v(\dot{x}) = \frac{\partial v(x,t)}{\partial t} = \left\{ \frac{\dot{\Delta}\left(\frac{-\frac{16*E*In*x^3}{L^3}+\frac{12*E*In*x^2}{L^2}}{E*In}\right)}{\dot{\Delta}\left(\frac{-\frac{16*E*In*x^3}{L^3}+\frac{32*E*In*\left(x-\frac{L}{2}\right)^3}{L^3}+\frac{12*E*In*x^2}{L^2}}{E*In}\right)} \right\}$$

The kinetic energy carried by the beam during vibration is:

$$K.E. = \int \frac{1}{2}\left[v(\dot{x})\right]^2 dm = \int \frac{1}{2}\left[v(\dot{x})\right]^2 \rho A dx$$

Thus

$$K.E. = \frac{\rho A\dot{\Delta}^2}{2}\left\{\int_0^{\frac{L}{2}}\left[\frac{4x^2(-4x+3L)}{L^3}\right]^2 + \int_{\frac{L}{2}}^{L}\left[-\frac{4(-4x^3+9x^2L-6xL^2+L^3)}{L^3}\right]^2\right\}$$

$$K.E. = \frac{\rho A\dot{\Delta}^2}{2}\left\{\frac{13}{70}L + \frac{83}{280}L\right\}$$

$$K.E. = \frac{\rho A\dot{\Delta}^2}{2}\{0.482\text{L}\}$$

Equate this kinetic energy with that of the equivalent system:

$$K.E = \frac{1}{2}m_e\dot{\Delta}^2$$

$$\frac{1}{2}m_e\dot{\Delta}^2 = \frac{\rho A\dot{\Delta}^2}{2}\{0.482\text{L}\} \rightarrow m_e = m_{beam}\{0.482\}$$

Appendix

These are random parameter for graph plot:

$L := 10$

$$10$$

$a := 3$

$$3$$

$F := 1000($

$$1000($$

$E := 200 \cdot 10^{9}$

$$20000000000$$

$In := 9 \cdot 10^{-9}$

$$\frac{9}{1000000000}$$

$$M_a := \frac{F\,a\,(L^2 - 2\,L\,a + a^2)}{L^2}$$

$$1470($$

$$M_b := \frac{a^2\,F\,(L-a)}{L^2}$$

$$630($$

$$Vshear := x \rightarrow piecewise\left(0 \le x < a, \frac{(-M_b + M_a - F \cdot a + F \cdot L)}{L}, a \le x \le L, \frac{(-M_b + M_a - F \cdot a)}{L}\right)$$

$$x \rightarrow piecewise\left(0 \le x \text{ and } x < a, \frac{-M_b + M_a - F\,a + F\,L}{L}, a \le x \text{ and } x \le L, \frac{-M_b + M_a - F\,a}{L}\right)$$

$$moment := x \rightarrow piecewise\left(0 \le x < a, \frac{1}{E \cdot In} \cdot \left(\frac{(-M_b + M_a - F \cdot a + F \cdot L)}{L} \cdot x - M_a\right), a \le x \le L, \frac{1}{E \cdot In} \cdot \left(\frac{(-M_b + M_a - F \cdot a + F \cdot L)}{L} \cdot x - F \cdot (x-a) - M_a\right)\right)$$

$$x \rightarrow piecewise\left(0 \le x \textbf{ and } x < a, \frac{\frac{\left(-M_b + M_a - F\,a + F\,L\right)x}{L} - M_a}{E\,In}, a \le x \textbf{ and } x \le L, \frac{\frac{\left(-M_b + M_a - F\,a + F\,L\right)x}{L} - F\,(x-a) - M_a}{E\,In}\right)$$

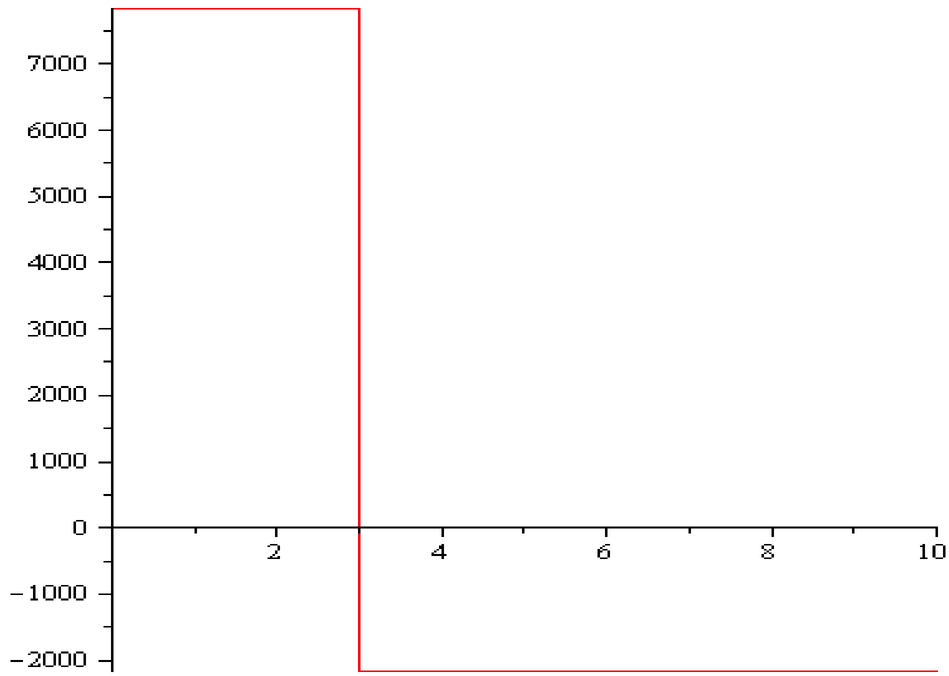
$$slope := x \rightarrow piecewise\left(0 \le x < a, \frac{1}{E \cdot In} \cdot \left(\frac{\left(-M_b + M_a - F \cdot a + F \cdot L\right)}{2 \cdot L} \cdot x^2 - M_a \cdot x\right), a \le x \le L, \frac{1}{E \cdot In} \cdot \left(\frac{\left(-M_b + M_a - F \cdot a + F \cdot L\right)}{2 \cdot L} \cdot x^2 - \frac{F}{2} \cdot (x-a)^2 - M_a \cdot x\right)\right)$$

$$x \rightarrow piecewise\left(0 \le x \textbf{ and } x < a, \frac{\frac{1}{2}\frac{\left(-M_b + M_a - F\,a + F\,L\right)x^2}{L} - M_a\,x}{E\,In}, a \le x \textbf{ and } x \le L, \frac{\frac{1}{2}\frac{\left(-M_b + M_a - F\,a + F\,L\right)x^2}{L} - \frac{1}{2}F\,(x-a)^2 - M_a\,x}{E\,In}\right)$$

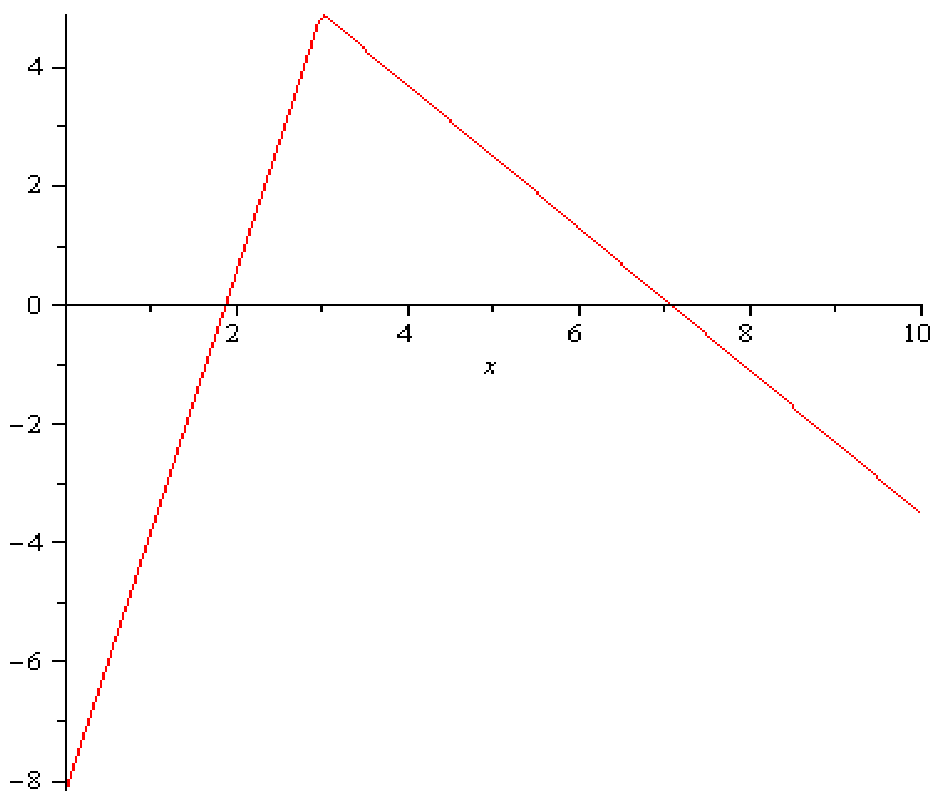
$$deflection := x \rightarrow piecewise\left(0 \le x < a, \frac{1}{E \cdot In} \cdot \left(\frac{\left(-M_b + M_a - F \cdot a + F \cdot L\right)}{6 \cdot L} \cdot x^3 - \frac{M_a}{2} \cdot x^2\right), a \le x \le L, \frac{1}{E \cdot In} \cdot \left(\frac{\left(-M_b + M_a - F \cdot a + F \cdot L\right)}{6 \cdot L} \cdot x^3 - \frac{F}{6} \cdot (x-a)^3 - \frac{M_a}{2} \cdot x^2\right)\right)$$

$$x \rightarrow piecewise\left(0 \le x \textbf{ and } x < a, \frac{\frac{1}{6}\frac{\left(-M_b + M_a - F\,a + F\,L\right)x^3}{L} - \frac{1}{2}M_a\,x^2}{E\,In}, a \le x \textbf{ and } x \le L, \frac{1}{E\,In}\left(\frac{1}{6}\frac{\left(-M_b + M_a - F\,a + F\,L\right)x^3}{L} - \frac{1}{6}F\,(x-a)^3 - \frac{1}{2}M_a\,x^2\right)\right)$$

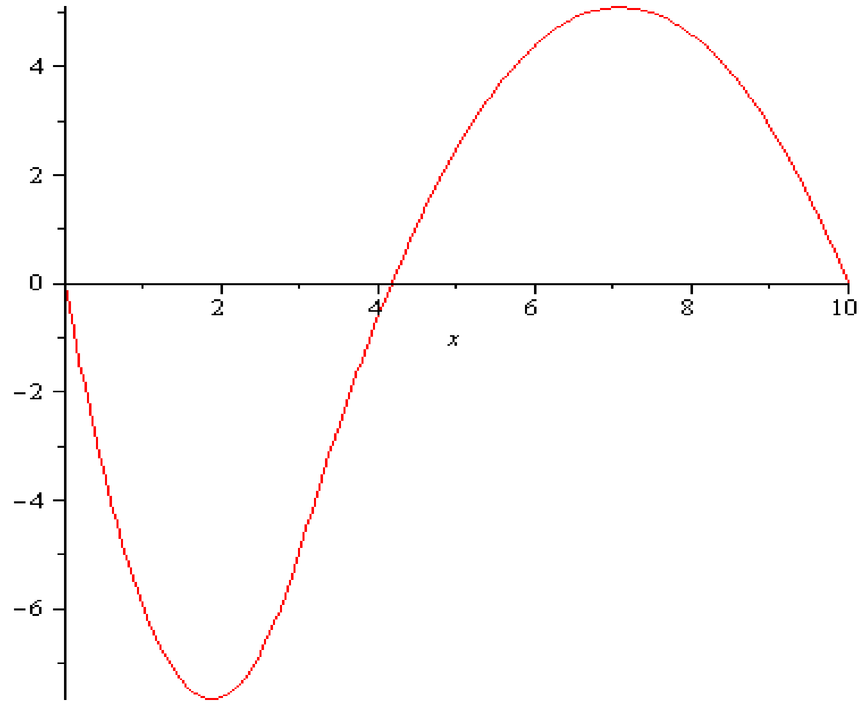
$plot(shear(x), x = 0..L)$



$plot(moment(x), x = 0..L)$



plot(slope(x), x = 0 ..L)



plot(deflection(x), x = 0 ..L)